Characterizing Delta-Matroid with Twist Monomials

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Contents

Background

Definition: A **Delta-Matroid** D is a set system $(E, \mathcal{F})$ such that $\mathcal{F}$ is a non-empty set that satisfies the symmetric exchange axiom, where: For any $X, Y \in \mathcal{F}$ and $u \in X \Delta Y$ there exists $v \in X \Delta Y$ such that $X \Delta \{u, v\} \in \mathcal{F}$.

Definition: Let $D = (E, \mathcal{F})$ be a delta-matroid. Then we can define:

$$\mathcal{F}_{max}(D) := \{F \in \mathcal{F} : F \text{ has maximum cardinality}\}$$

$$\mathcal{F}_{min}(D) := \{F \in \mathcal{F} : F \text{ has minimum cardinality}\}$$

The **upper matroid of** D is defined to be $D_{max} = (E, \mathcal{F}_{max})$, and the **lower matroid of** D is $D_{min} = (E, \mathcal{F}_{min})$.

Note that since D_{max}/D_{min} are matroids we can assign a rank function r_D

Background

Definition: The **width of** D is defined to be:

$$w(D) := r(D_{max}) - r(D_{min})$$

Definition: Let $D = (E, \mathcal{F})$ be a delta matroid, and $A \subseteq E$ the **twist of** D with A is defined as:

$$D * A = (E, \{F \Delta A : F \in \mathcal{F}\})$$

Definition: Let $D = (E, \mathcal{F})$ be a delta matroid, the **Twist Polynomial of** D is given by:

$$^{\partial}w_D(z) := \sum_{A \subseteq E} z^{w(D*A)}$$

Background

Definition: A delta-matroid D is **even** if the cardinality of all $F \in \mathcal{F}$ have the same parity. D is **odd** if it is not even.

Let E be a finite set, and C be a symmetric $|E| \times |E|$ matrix over GF(2). Where the rows and columns of C are indexed by elements from E. Now for any $A \subseteq E$ we can define $C[A]$ to be a sub-matrix of C given by the elements of A.

Then we can note that the set system $D(C) = (E, \mathcal{F})$ where:

$$\mathcal{F} := \{A \subseteq E : C[A] \text{ is non-singular}\}$$

is a normal delta-matroid (as $C[\emptyset]$ is considered non-singular).

Definition: A delta-matroid D is **binary** if there exists a twist isomorphic to $D(C)$ for some symmetric matrix over $GF(2)$.

Background

If $D = (E, \mathcal{F})$ is a normal binary delta-matroid there exists a unique symmetric $|E| \times |E|$ matrix where $D = D(C)$. This matrix can be construct via the following process:

- Put $C_{vv} = 1$ iff $\{v\} \in \mathcal{F}$
- Put $C_{uv} = 1$ iff $\{u\}, \{v\} \in \mathcal{F}$ and $\{u, v\} \notin \mathcal{F}$, **OR** both $\{u\}, \{v\}$ are not in $\mathcal{F}$ and $\{u, v\} \in \mathcal{F}$,

Definition: The **intersection graph of** D, G_D is the graph with vertex set E and $u, v \in E$ are adjacent if and only if $C_{uv} = 1$

Definition: Let $D = (E, \mathcal{F})$ be a delta-matroid. Let $e \in E$ Then e is a **ribbon loop** if e is in no feasible set of D_{min}.
A ribbon loop e is **non-orientable** if e is a ribbon loop in $D * e$ and it is **orientable** otherwise.

Background

Definition: Let $D = (E, \mathcal{F})$ be a delta-martoid, and let $e \in E$. Then the deletion of e, $D \backslash e$, is defined to be:

$$D\backslash e := \begin{cases} (E, \{F\backslash e, F \in \mathcal{F}\}) & \text{if for any } F \in \mathcal{F},\ e \in F \\ (E, \{F : F \in \mathcal{F}, F \subseteq E\backslash e\}) & \text{else} \end{cases}$$

Definition: Let $D = (E, \mathcal{F})$ be a delta-martoid, and let $e \in E$. Then the contraction of e, D/e, is defined to be:

$$D/e := \begin{cases} (E, \{F\backslash e, F \in \mathcal{F}, e \in F\}) & \text{if for any } F \in \mathcal{F},\ e \notin F \\ (E, \mathcal{F}) & \text{else} \end{cases}$$

We can note that we have this identity, $D^*/e = (D\backslash e)^*$.

Twist Monomials

The goal of this presentation is to characterize twist monomials and their associated delta-matroid.

Twist Monomials

Theorem [Yan, Jin, 2021]
Let $D = (E, \mathcal{F})$ be a connected normal binary delta-matroid. Then ${}^{\partial}w_D(z) = mz^k$ if and only if each component of G_D is either a complete graph of odd order, or a single vertex with a loop.

Twist Monomials

To prove the previous theorem, the main approach will be to break the proof into two main cases which will be if the delta-matroid is even or odd.

Odd Case

Theorem: Let $D = (E, \mathcal{F})$ be a connected odd normal binary delta-matroid. Then ${}^{\partial}w_D(z) = mz^k$ if and only if $D = (\{1\}, \{\emptyset, \{1\}\})$.

Proof: ($\rightarrow$) If D=({1}, $\{\emptyset, \{1\}\}$) then we have:

$$ {}^{\partial}w_D(z) = 2z $$

($\leftarrow$) We now need to prove that $|E| = 1$, so assume this is not true then:

Claim 1: For any $e, f \in E$, $D_{|\{e,f\}} \neq (\{e, f\}, \{\emptyset, \{e\}, \{e, f\}\})$

Now to prove this claim holds, we will continue via contradiction.

Odd Case

Since we have the assumption that ${}^{\partial}w_D(z) = mz^k$ we know that:

$$w(D) = w(D * e) = w(D * f)$$

We can then note that in $D^* = D * E$ that e will be a non-orentable ribbon loop, and f will be a non-ribbon loop. We also know that f is a non ribbon loop of D^*/e.

Odd Case

Since f is a non-ribbon loop of D^*/e we have:

$$F \in \mathcal{F}_{min}(D^*/e), \text{ where } f \in F$$

Now define

$$F' := (E \backslash e) \backslash F)$$

we can note that $e, f \notin F'$ and that F' is in both $\mathcal{F}_{max}(D \backslash e)$ and $\mathcal{F}(D)$. Now for any $X \in \mathcal{F}_{max}(D)$ we will always have $e \in X$, because if not

$$w(D * e) = w(D) + 1$$

which forms a contradiction. Now we will always have,

$$r((D \backslash e)_{max}) \leq r(D_{max}) - 1$$

Odd Case

Now if for any $Y \in \mathcal{F}(D)$ such that $|Y| = r(D_{max}) - 1$ we always have $e \in Y$ it follows

$$r(D * e_{max}) = r(D_{max}) - 1$$

So we have a contradiction, so there exists Y such that $e \notin Y$ and $|Y| = r(D_{max}) - 1.$, and $Y \in D \backslash e$, so:

$$r((D \backslash e)_{max}) = r(D_{max}) - 1$$

Now recall F' we get that:

$$|F'| = r(D_{max}) - 1$$

so

$$w(D * \{e, f\}) \geq |F' \cup \{e, f\}| = w(D) + 1$$

Odd Case

Claim 2: For any $e, f \in E$, $D_{|\{e,f\}} \neq (\{e, f\}, \{\emptyset, \{e\}, \{f\}\}$

Assume that this claim is not true, we have:

$$D * e_{|\{e,f\}} = (\{e, f\}, \{\emptyset, \{e\}, \{e, f\}\})$$

Lemma: Let $D = (E, \mathcal{F})$ be a delta-matroid, with $A \subseteq E$, then:

$$^{\partial}w_D(z) = {}^{\partial}w_{D*A}(z)$$

Thus we get our contradiction:

$$^{\partial}w_{D*e}(z) = {}^{\partial}w_D(z)$$

Odd Case

Now since D is a normal binary delta-matroid there exists a symmetric matrix C, where $D = D(C)$. Then since D is connected, there exists $f \in E$ such that:

$$C[\{e, f\}] = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

or

$$C]\{e, f\}] = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

which both matrices correspond the delta-matroids in claim 1 and claim 2.

Corollary: Let $D = (E, \mathcal{F})$ be a connected odd normal binary delta-matroid. Then $^{\partial}w_D(z) = mz^k$ iff G_D is a single vertex with a loop.

Even Case

Theorem: Let $D = (E, \mathcal{F})$ be a connected even normal binary delta-matroid. Then ${}^{\partial}w_D(z) = mz^k$ if and only if G_D is a complete graph of odd order.

Proof: $(\rightarrow)$ We can assume that G_D is a complete graph of odd order, to prove this direction we need to prove this proposition below:

Proposition: Let D be a normal binary delta-matroid, and let v be the number of vertices of G_D, if G_D is a complete graph then:

$$ {}^{\partial}w_D(z) = \begin{cases} 2^v z^{v-1} & \text{if } v \text{ is odd,} \\ 2^{v-1} z^v + 2^{v-1} z^{v-2} & \text{if } v \text{ is even} \end{cases} $$

Even Case

Proof of Proposition: Let B_v be a bouquet with signed rotation $(1, 2, 3, ..., v, 1, 2, 3, ..., v)$ such that:

$$G_{D(B_v)} = G_D = K_V$$

Definition: The **partial Dual Euler genus polynomial** for a G is defined to be:

$$^{\partial}\varepsilon_G(z) = \sum_{A \subseteq E(G)} z^{\varepsilon(G^A)}$$

where $\varepsilon(G) = 2c(G) - \chi(G)$

Now via a few theorems we have:

$$^{\partial}w_D(z) = {}^{\partial}w_{D(B_V)}(z) = {}^{\partial}\varepsilon_{B_v}(z)$$

Even Case

Then we can note that for a bouquet B_v we have:

$$^{\partial}\varepsilon_{B_v}(z) = \begin{cases} 2^v z^{v-1} & \text{if } v \text{ is odd,} \\ 2^{v-1}z^v + 2^{v-1}z^{v-2} & \text{if } v \text{ is even.} \end{cases}$$

($\leftarrow$) Now assumes that $^{\partial}w_D(z) = mz^k$, for $|E| \in \{1, 2\}$ this claim is easily verifiable as we have:

$$D = \begin{cases} (\{e\}, \{\emptyset\}) & \text{if } |E| = 1, \\ (\{e.f\}.\{\emptyset, \{e, f\}\} & \text{if } |E| = 2. \end{cases}$$

For $|E| \geq 3$ with $e, f, g \in E$ we will consider three different claims.

Even Case

Definition: Let $M = (E, \mathcal{B})$ be a matroid, for any $A \subseteq E$ the **rank function of** M is defined as:

$$r_M(A) = max\{A \cap B : B \in \mathcal{B}\}$$

Even Case

Claim 1: If $\{e, f\} \in \mathcal{F}$, we have $r_{D^*_{min}}(\{e, f\}) = 1$ and $r_{D^*_{min}}(\{e\}) = 1$.

We can note that for any $A \in \mathcal{F}_{max}(D)$ we have $A \cap \{e, f\} \neq \emptyset$, as if this were not true, $w(D * \{e, f\}) > w(D)$.

Additionally there exists a $B \in \mathcal{F}_{max}(D)$ such that $e \in B$, because otherwise $w(D * e) = w(D) - 2$

This then implies that $A \in \mathcal{F}_{min}(D^*)$, $A \cap \{e, f\} \neq \{e, f\}$ and that there exists $B \in \mathcal{F}_{min}(D^*)$ with $e \in B$ which proves our claim.

Even Case

Claim 2: If $\{e, f\} \notin \mathcal{F}(D)$, then $r_{D^*_{min}}(\{e, f\}) = 2$.

There exists an $A \in \mathcal{F}_{max}(D)$ where $A \cap \{e, f\} = \emptyset$, otherwise we would have:

$$w(D * \{e, f\}) \leq w(D) - 2$$

This then implies that $A \in \mathcal{F}_{min}(D^*)$ where $\{e, f\} \in A$ and $r_{D^*_{min}}(\{e, f\}) = 2$.

Even Case

Claim 3: Our collection of feasible sets $\mathcal{F}$ is not constructed in a way where $\{e, f\}, \{e, g\} \in \mathcal{F}$ and $\{f, g\} \notin \mathcal{F}$.

Assume this claim is not true, via Claim 1 we have that $r_{D^*_{min}}(\{e, f\}) = r_{D^*_{min}}(\{e, g\}) = 1$, which gives us:

$$\begin{aligned} & 2r_{D^*_{min}}(\{e, f\} \cup \{e, g\}) \\ &= r_{D^*_{min}}(\{e, f, g\}) + r_{D^*_{min}}(\{e\}) \\ &\leq r_{D^*_{min}}(\{e, f\}) + r_{D^*_{min}}(\{e, g\}) \\ &= 2 \end{aligned}$$

which then forms a contradiction as:

$$2 = r_{D^*_{min}}(\{f, g\}) \leq r_{D^*_{min}}(\{e, f, g\}) \leq 1$$

Even Case

Now for the last step we can assume that G_D is not a complete graph, which then the symmetric matrix C that corresponds with out delta-matroid to take the form:

$$C[\{e, f, g\}] = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

This then implies that $\{e, f\}, \{e, g\} \in \mathcal{F}$ and $\{f, g\} \notin \mathcal{F}$, thus G_D must be complete. Now we can generalize this theorem via the following corollary:

> **Corollary:** Let $D = (E, \mathcal{F})$ be an even normal binary delta-matroid. Then ${}^{\partial}w_D(z) = mz^k$ if and only if each connected component of G_D is a complete graph of odd order.

Twist Monomials

Now we can combine the two previous theorems that we proved to get us:

Theorem [Yan, Jin, 2021]
Let $D = (E, \mathcal{F})$ be a connected normal binary delta-matroid. Then ${}^{\partial}w_D(z) = mz^k$ if and only if each component of G_D is either a complete graph of odd order, or a single vertex with a loop.

Another Theorem

Theorem [Yus, 2024]
Let $D = (E, \mathcal{F})$ be a delta-matroid. If ${}^{\partial}w_D(z) = mz^k$, then D is binary.

Sketch of Proof

We will assume that $D = (E, \mathcal{F})$ is a non-binary delta-matroid where ${}^{\partial}w_D(z) = mz^k$.

Since D is non-binary, there exists a minor of D (which is derived from a series of twist, contractions, and deletions) that is isomorphic to:

$$D_1 = (\{1, 2\}, \{\emptyset, \{1\}, \{2\}\})$$

$$D_2 = (\{1, 2, 3\}, \{\emptyset, \{1, 2\}, \{1, 3\}\})$$

Since D_1 or D_2 is normal there exists $F \in \mathcal{F}$ and $A \subseteq E$ where $D_i \cong (D * F)\backslash A$.

But $D * F$ is normal, so no restriction is isomorphic to D_1/D_2.

Characterization

Utilizing both major theorems we can come to characterize all normal delta-matroid who have twist monomials.

Definition: The direct sum of delta-matroids $D = (E, \mathcal{F})$ and $\tilde{D} = (\tilde{E}, \tilde{\mathcal{F}})$ with $E \cap \tilde{E} = \emptyset$ is defined as:

$$D \oplus \tilde{D} := (E \cup \tilde{E}, \{F \cup \tilde{F} : F \in \mathcal{F},\ \tilde{F} \in \tilde{\mathcal{F}}\}).$$

Now we can define a delta-matroid $N_1 = (\{1\}, \{\emptyset, \{1\}\})$, where $N_n := \bigoplus_n N_1$, getting us:

$$^{\partial}w_{N_n}(z) = 2^n z^n$$

Then let D be an even normal delta-matroid with $^{\partial}w_D(z) = mz^k$, then all normal delta-matroids with twist monomials are of the form:

$$D \oplus N_n$$

References

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Thank you!

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